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Another Proof of a Theorem Concerning Detachments of Graphs

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Let b be a positive integer-valued function on the set of vertices of a finite graph G. We give a new proof of a theorem which characterizes the least possible number of components of a graph obtainable from G by splitting each vertex ξ into $b(\xi)$ vertices.

In this paper, graphs are understood to be finite, and may have loops and multiple edges. The letter G denotes a graph, b denotes a function from $V(G)$ into the set of positive integers and $c(G)$ denotes the number of components of G. For X, $Y \subseteq V(G)$, let $X \nabla_G Y$ be the set of those edges of G which join an element of X to an element of Y. Let

$$\tilde{X} = V(G)\backslash X, \qquad b \cdot X = \sum_{\xi \in X} b(\xi),$$

$$\hat{L}(X) = X \nabla_G X, \qquad L(X) = X \nabla_G V(G)$$

and let $G - X$ denote the subgraph of G such that $V(G-X) = \tilde{X}$, $E(G-X) = \hat{L}(\tilde{X})$. If $\xi \in V(G)$ then $L(\xi)$ means $L(\{\xi\})$. If $K \subseteq E(G)$ then $G \times K$ denotes the subgraph of G such that $V(G \times K) = V(G)$, $E(G \times K) = K$. If λ is an edge of a graph F then clearly $c(F - \lambda) \leqslant c(F) + 1$: repeated use of this fact gives

$$c(G \times K) + |K \backslash J| \geqslant c(G \times (J \cap K)) \geqslant c(G \times J) \tag{1}$$

for all subsets J, K of $E(G)$.

For each $\xi \in V(G)$ let $\Omega\xi$ be a set of cardinality $b(\xi)$, such that the sets $\Omega\xi$ $(\xi \in V(G))$, $E(G)$ are disjoint. Let D be a graph such that $V(D) = \bigcup_{\xi \in V(G)} \Omega\xi$, $E(D) = E(G)$ and $\Omega\xi \nabla_D \Omega\eta = \{\xi\} \nabla_G \{\eta\}$ for all ξ, $\eta \in V(G)$. Then we shall say that D is a *b-detachment* of G. Intuitively, D is obtained from G by splitting each vertex ξ into the elements of $\Omega\xi$ and so a b-detachment of G is obtained from G by splitting each vertex ξ of G into $b(\xi)$ vertices. The class of all b-detachments of G will be denoted by $\mathscr{D}(G, b)$.

The first half of [2, Theorem 3] states that

$$\min_{D \in \mathscr{D}(G,b)} c(D) = \max_{X \subseteq V(G)} (b \cdot X + c(G - X) - |L(X)|). \tag{2}$$

Denoting the left- and right-hand sides of (2) by m and M respectively, the non-trivial part of the proof of (2) is the proof that $m \leqslant M$, which was deduced from a theorem about matroids in [2]. The purpose of this note is to give another proof that $m \leqslant M$, again using matroids but in a different way.

For any required background concerning matroids, see [3]. We mention only that $(S, \mathscr{I})$ denotes a matroid the underlying set of which is S, and the set of independent sets of which is $\mathscr{I}$; and that the *matroid of G* is the matroid $(E(G), \mathscr{I}_G)$, where a subset of $E(G)$ belongs to $\mathscr{I}_G$ iff it is the set of edges of a subforest of G.

We require the following well known theorem, implicit in [1]: a proof and further background may also be found in [3, Ch. 8].

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THEOREM. *Let $(S, \mathscr{I}_1), \ldots, (S, \mathscr{I}_n)$ be matroids with the same underlying set S. Then there exist disjoint sets $I_1 \in \mathscr{I}_1, \ldots, I_n \in \mathscr{I}_n$ such that*

$$|I_1 \cup \cdots \cup I_n| = \min_{A \subseteq S} \Big(\sum_{j=1}^{n} r_j(A) + |S \backslash A| \Big), \tag{3}$$

where r_j denotes rank in $(S, \mathscr{I}_j)$.

[More specifically, the sets

$$I_1 \cup \cdots \cup I_n \qquad (I_1 \in \mathscr{I}_1, \ldots, I_n \in \mathscr{I}_n)$$

are the independent sets of a matroid the rank of which is the right-hand side of (3), but we do not need this fact.]

Let $V(G) = \{\xi_1, \ldots, \xi_n\}$. Let $\mathscr{I}_0 = \mathscr{I}_G$ and for $j = 1, \ldots, n$ let $\mathscr{I}_j$ be the set of all sets of $b(\xi_j) - 1$ or fewer elements of $L(\xi_j)$. Then, by the above theorem, there exist disjoint sets $I_0 \in \mathscr{I}_0$, $I_1 \in \mathscr{I}_1, \ldots, I_n \in \mathscr{I}_n$ and a set $A \subseteq E(G)$ such that

$$|I_0 \cup I_1 \cup \cdots \cup I_n| = \sum_{j=0}^{n} r_j(A) + |E(G) \backslash A|, \tag{4}$$

where r_j denotes rank in the matroid $(E(G), \mathscr{I}_j)$ for $j = 0, 1, \ldots, n$. Let

$$Y = \{\xi \in V(G): |L(\xi) \cap A| < b(\xi) - 1\}.$$

Then for $j = 1, \ldots, n$ we have $r_j(A) = |L(\xi_j) \cap A|$ if $\xi_j \in Y$ and $r_j(A) = b(\xi_j) - 1$ if $\xi_j \in \tilde{Y}$. Moreover, $r_0(A) = |V(G)| - c(G \times A)$. These observations, together with (4) and (1), give

$$\begin{aligned}
|I_0 \cup I_1 \cup \cdots \cup I_n| &= |V(G)| - c(G \times A) + \sum_{\xi \in Y} |L(\xi) \cap A| \\
&\quad + \sum_{\xi \in \tilde{Y}} (b(\xi) - 1) + |E(G) \backslash A| \\
&\geqslant |V(G)| - c(G \times \hat{L}(\tilde{Y})) - |\hat{L}(\tilde{Y}) \backslash A| \\
&\quad + |L(Y) \cap A| + b \cdot \tilde{Y} - |\tilde{Y}| + |E(G) \backslash A| \\
&= |V(G)| - (|Y| + c(G - Y)) + |L(Y)| + b \cdot \tilde{Y} - |\tilde{Y}| \\
&= b \cdot V(G) - (b \cdot Y + c(G - Y) - |L(Y)|) \geqslant b \cdot V(G) - M.
\end{aligned}$$

Let $\Omega\xi_1, \ldots, \Omega\xi_n$ be disjoint sets such that $\xi_j \in \Omega\xi_j$, $\Omega\xi_j \cap E(G) = \varnothing$ and $|\Omega\xi_j| = b(\xi_j)$ for $j = 1, \ldots, n$. Let f_j be an injection of I_j into $\Omega\xi_j \backslash \{\xi_j\}$ for $j = 1, \ldots, n$. If $j \in \{1, \ldots, n\}$ and $\lambda \in I_j$ let $g_j(\lambda)$ be the vertex to which λ joins ξ_j in G. (In particular, $g_j(\lambda) = \xi_j$ if λ is a loop.) Let D be the b-detachment of G such that:

(i) $V(D) = \Omega\xi_1 \cup \cdots \cup \Omega\xi_n$, $E(D) = E(G)$;

(ii) if $j \in \{1, \ldots, n\}$ and $\lambda \in I_j$ then λ joins $f_j(\lambda)$ to $g_j(\lambda)$ in D;

(iii) each element of $E(G) \backslash (I_1 \cup \cdots \cup I_n)$ joins the same vertices in D as in G.

Since $I_0 \in \mathscr{I}_0$ it follows that $c(G \times I_0) = |V(G)| - |I_0|$. Moreover, $G \times I_0$ is a subgraph of D and each element of $f_1(I_1) \cup \cdots \cup f_n(I_n)$ is joined by an edge of D to a vertex of G. Therefore

$$\begin{aligned}
c(D) &\leqslant c(G \times I_0) + |V(D) \backslash (V(G) \cup f_1(I_1) \cup \cdots \cup f_n(I_n))| \\
&= (|V(G)| - |I_0|) + (b \cdot V(G) - |V(G)| - |I_1| - \cdots - |I_n|) \\
&= b \cdot V(G) - |I_0 \cup I_1 \cup \cdots \cup I_n| \leqslant M.
\end{aligned}$$

Since $D \in \mathscr{D}(G, b)$ it follows that $m \leqslant M$, as required.

References

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